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Change in Skew Distance Energy due to edge deletion

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Abstract-The skew distance energy of a directed graph G^{ϕ} is the sum of the absolute values of all skew distance eigen values of the skew distance matrix of G^{ϕ}. In this paper we present how the skew distance energy of G^{ϕ} changes when edges are deleted. Examples show that all cases are possible : increased, decreased, unchanged. Our aim is to find possible graph theoretical descriptions and to provide an infinite family of graphs for each case.

Keywords-Energy of a graph; distance energy; oriented graphs; Skew energy; Skew distance energy; Change in Skew Distance Energy.

1. INTRODUCTION

Let $G(V,E)$ be a finite simple connected graph with n vertices and m edges. Let G^{ϕ} be a directed graph with an orientation ϕ. A subgraph H of G is an induced subgraph of G if H contains all edges of G that join two vertices of H. The complement of H in G is denoted as $G\backslash H$ and is obtained from G by deleting all vertices of G an induced subgraph H and all edges incident with H. Moreover when no edge of G join H and its complement $G\backslash H$, we write $G = H \oplus (G\backslash H)$. Let λ_i, $i = 1,2,\ldots,n$ be the skew distance eigen values of the skew distance matrix of a directed graph G^{ϕ}. The skew distance energy of G^{ϕ} is denoted as $E_{SD}(G^{\phi})$ and is defined as

$$E_{SD}(G^{\phi}) = \sum_{i=1}^{n} |\lambda_i| (5)$$

The singular values of $A \in M_{n\times m}(C)$ are the square roots of the eigen values of $A\bar{A}^T$ which are denoted by $S_1(A) \geq S_2(A) \geq \ldots \geq S_n(A) \geq 0$.

Nikiforov [3] defined the energy of any $A \in M_{n\times m}(C)$ as $\sum_{i=1}^{n} S_i(A)$ which coincides with the definition of energy $\sum_{i=1}^{n} |\lambda_i(A)|$ if and only if A is a normal matrix (see section 4 in [4]). As the skew distance matrix of G^{ϕ} is real skew symmetric and the real skew symmetric matrices are normal $(A\bar{A}^T = \bar{A}^T A)$, the skew distance matrix of G^{ϕ} is normal. Therefore $\sum_{i=1}^{n} S_i(A) = \sum_{i=1}^{n} |\lambda_i(A)|$, A is the skew distance matrix of G^{ϕ}.

In Section 2, we prove a singular value inequality for complementary submatrices and characterize the equality case. Then this inequality is applied in section 3 to obtain results in skew distance energy change when a cutset is deleted. Section 4 presents several infinite families of digraphs, each having an interesting skew distance energy property when an edge is deleted.

2. A SINGULAR VALUE INEQUALITY

Lemma 2.1 *[1]Let C be a complex $m \times n$ matrix. Then $|\sum_{i=1}^{n} |\lambda_i(C)| \leq \sum_{i=1}^{n} S_i(C)$. Equality holds iff there exists a real scalar θ such that $e^{i\theta}C$ is positive and semi-definite.*

Theorem 2.2 *[1]For a partitioned matrix $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$, where both A and B are square matrices, we have*

$$\sum_j S_j(A) + \sum_j S_j(B) \leq \sum_j S_j(C).$$

Equality holds if and only if there exists unitary matrices U and V such that $\begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix}$ is positive semi-definite.

Theorem 2.3 *For a partitioned skew distance matrix of G^{ϕ}, $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$, where both A and B are square matrices and $Y = -X^T$, we have*

$$\sum_i |\lambda_i(A)| + \sum_i |\lambda_i(B)| \leq \sum_i |\lambda_i(C)|.$$

Equality holds if and only if there exists unitary matrices U and V such that $\begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix}$ is positive semi-definite.

Proof. As the skew distance matrix of G^{ϕ} is real skew symmetric and the real skew symmetric matrices are normal, C- the skew distance matrix of G^{ϕ} is normal $(C\bar{C}^T = \bar{C}^T C)$. Therefore,

$$\sum_{i=1}^{n} S_i(C) = \sum_{i=1}^{n} |\lambda_i(C)|$$

,

C is the skew distance matrix of G^{ϕ}. By theorem 2.2 [1],

$$\sum_i S_i(A) + \sum_i S_i(B) \leq \sum_i S_i(C).$$

Therefore, $\sum_i |\lambda_i(A)| + \sum_i |\lambda_i(B)| \leq \sum_i |\lambda_i(C)|$, which proves the inequality. Moreover, equality holds if and only if

$$traceC' = \sum_{i=1}^{n} S_i(C') = \sum_{i=1}^{n} S_i(C),$$

$where\ C' =$ $\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} C$;UandVareunitarymatricessuchthatUAandVBarepositivesemi-definite, if and only if there exists θ such that $e^{i\theta}C'$ is positive semi-definite, by lemma 2.1 [1]. Since $traceC'$ is non negative , $e^{i\theta} = 1$, (i.e) C' is positive semi-definite. Hence equality holds if and only if there exist unitary matrices U and V such that $\begin{bmatrix} UA & UX \\ -VX^T & VB \end{bmatrix}$ is positive semi-definite.

Corollary 2.4 *[1]For a partitioned matrix* $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$, *where* A *and* B *are square matrices and* $Y = -X^T$, *we have* $\sum_i S_i(A) \le \sum_i S_i(C)$. *Equality holds if and only if* X, Y *and* B *are all zero matrices.*

Corollary 2.5 *For a partitioned skew distance matrix of* G^{ϕ}, $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$, *where* A *and* B *are square matrices and* $Y = -X^T$, *we have* $\sum_i |\lambda_i(A)| \le \sum_i |\lambda_i(C)|$. *Equality holds if and only if* X, Y *and* B *are all zero matrices.*

Proof. From theorem 2.2 [1],
$\sum_i S_i(A) + \sum_i S_i(B) \le \sum_i S_i(C)$.
(i.e)$\sum_i S_i(A) \le \sum_i S_i(A) + \sum_i S_i(B) \le \sum_i S_i(C)$
(i.e)$\sum_i S_i(A) \le \sum_i S_i(C)$.
Therefore
$\sum_i |\lambda_i(A)| \le \sum_i |\lambda_i(C)|$.
Assume that X, Y and B are all zero matrices. Then $C = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$.
Clearly, $\sum_i S_i(C) = \sum_i S_i(A)$. i.e. $\sum_i |\lambda_i(C)| = \sum_i |\lambda_i(A)|$.
Conversely, let us assume that $\sum_i |\lambda_i(A)| = \sum_i |\lambda_i(C)|$. Then $\sum_i S_i(A) = \sum_i S_i(C)$. As

$$\sum_i S_i(A) = \sum_i S_i(A) + \sum_i S_i(B) = \sum_i S_i(C),$$

$\sum_i S_i(B) = 0$ and so $B = 0 \quad (zeromatrix)$.
Moreover the equality$\sum_i S_i(A) + \sum_i S_i(B) = \sum_i S_i(C)$ implies that there exist unitary matrices U and V such that $\begin{bmatrix} UA & UX \\ VY & VB \end{bmatrix} = \begin{bmatrix} UA & UX \\ VY & 0 \end{bmatrix}$ is positive semi-definite where $Y = -X^T$. Consequently, $UX = 0$ and $UY = 0$ [[6], section 7.1, 7.1.P2], so $X = 0$ and $Y = 0$.

3. 3.EDGE SET DELETION

Theorem 3.1 *[1]Let* G' *be an induced subgraph of a simple graph* G. *Then the energy of* $\varepsilon(G') \le \varepsilon(G)$*-the energy of* G *and equality holds if and only if* $E(G') = E(G)$ *where* $E(G)$ *represents the edge set of* G.

Theorem 3.2 *Let* H^{ϕ} *be an induced subgraph of a directed graph* G^{ϕ}. *Then* $E_{SD}(H^{\phi}) \le E_{SD}(G^{\phi})$ *and equality holds if and only if Edge set of* H^{ϕ}*=Edge set of* G^{ϕ}.

Proof. The skew distance matrix of G^{ϕ} is $SD(G^{\phi}) = \begin{bmatrix} SD(H^{\phi}) & X \\ -X^T & SD(G^{\phi}\backslash H^{\phi}) \end{bmatrix}$. where the skew distance matrix X represents the edges that connect $G^{\phi}\backslash H^{\phi}$ and H^{ϕ}. By corollary 2.5, $E_{SD}(H^{\phi}) \le E_{SD}(G^{\phi})$.
For the equality case, suppose the edge set of H^{ϕ}= the edge set of G^{ϕ}, then $SD(G^{\phi}) = \begin{bmatrix} SD(H^{\phi}) & 0 \\ 0 & 0 \end{bmatrix}$.
Therefore $E_{SD}(G^{\phi}) = E_{SD}(H^{\phi})$.
Conversely, suppose that $E_{SD}(G^{\phi}) = E_{SD}(H^{\phi})$, then

$$\sum_{i=1}^{n} |\lambda_i(SD(G^{\phi}))| = \sum_{i=1}^{n} |\lambda_i(SD(H^{\phi}))|.$$

By corollary 2.5, X and $SD(G^{\phi}\backslash H^{\phi})$ are zero matrices. Thus the edge set of H^{ϕ} = the edge set of G^{ϕ}.

Corollary 3.3 *[2]For any simple graph* G *with atleast one edge,* $DE(G) \ge 2$

Corollary 3.4 *For any simple graph* G *with atleast one edge and with orientation* ϕ, $E_{SD}(G^{\phi}) \ge 2$.

Proof. Let $H^{\phi}: uv = {k_2}^{\phi}$ be an edge of G^{ϕ}. Then by theorem 3.2,
$E_{SD}(H^{\phi}) \le E_{SD}(G^{\phi})$.
Since $E_{SD}(H^{\phi}) = 2$, $E_{SD}(G^{\phi}) \ge 2$.

Definition 3.5 *If* E *is a set of edges of* G *such that* $G\backslash E$ *is the union of two complementary induced subgraph, then* E *is called a cutset of* G.

Theorem 3.6 *[2]If* E *is a cutset of a simple graph* G, *then* $DE(G\backslash E) \le DE(G)$.

Theorem 3.7 *If* E *is a cutset of a digraph* G^{ϕ}, *then* $E_{SD}(G^{\phi}\backslash E) \le E_{SD}(G^{\phi})$.

Proof. As E is a cutset of a digraph G^{ϕ}, $(G^{\phi}\backslash E) = H^{\phi} \oplus K^{\phi}$, where H^{ϕ} and K^{ϕ} are two complementary induced subgraph of G^{ϕ}. Then

$$SD(G^{\phi}) = \begin{bmatrix} SD(H^{\phi}) & X \\ -X^T & SD(K^{\phi}) \end{bmatrix}$$

where the skew distance matrix X represents the edges that connect H^{ϕ} and G^{ϕ}. Therefore by theorem 2.3,

$$\sum_i |\lambda_i(SD(H^{\phi}))| + \sum_i |\lambda_i(SD(K^{\phi}))| \le \sum_i |\lambda_i(SD(G^{\phi}))|.$$

Since $(G\backslash E)^{\phi}$ is the union of two complementary induced subgraphs H^{ϕ} and K^{ϕ},

$$E_{SD}\left[(G^{\phi}\backslash E)\right] = E_{SD}(H^{\phi}) + E_{SD}(K^{\phi}).$$

Therefore,

$$E_{SD}\left[(G^{\phi}\backslash E)\right] \le E_{SD}(G^{\phi}).$$

Theorem 3.8 *If* $\{e\}$ *is a bridge in a simple digraph* G^{ϕ}, *then* $E_{SD}(G^{\phi}\backslash\{e\}) \le E_{SD}(G^{\phi})$.

Proof. Take $E = \{e\}$ in theorem 3.7.

Corollary 3.9 *Let* e *be an edge of an oriented tree* T^{ϕ}. *Then*

$$E_{SD}(T^{\phi}\backslash\{e\}) < E_{SD}(T^{\phi})$$

.

Example 3.10 *Let* ${K_n}^{\psi}$ *be the oriented complete graph on* n *vertices with the orientations of all the arcs go from low labels to high labels. Then the skew distance energy of* ${K_n}^{\phi}$

is

$$\sum_{k=1}^{n} cot(2k-1)\frac{\pi}{2n}$$

Proof. As

$$SD(K_n{}^{\psi}) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & 1 & \cdots & 1 & 1 \\ -1 & -1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & 0 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{bmatrix}$$

$$|SD(K_n{}^{\psi}) - xI| = \begin{vmatrix} -x & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -x & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & -x & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -x & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -x & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & -x \end{vmatrix}$$

By 1.2. P_{16} in [6],

$$\begin{vmatrix} d_1 & b & b & \cdots & b & b \\ c & d_2 & b & \cdots & b & b \\ c & c & d_3 & \cdots & b & b \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ c & c & c & \cdots & d_{n-1} & b \\ c & c & c & \cdots & c & d_n \end{vmatrix} = \frac{bq(c) - cq(b)}{b-c} \, if b \neq c,$$

where $q(t) = (d_1 - t)(d_2 - t) \cdots (d_n - t)$.

Therefore, $|SD(K_n{}^{\psi}) - xI| = \frac{1q(-1)-(-1)q(1)}{1-(-1)}$ where

$q(t) = (-x-t)(-x-t) \ldots (-x-t)$
$= (-1)^n (x+t)^n$

Therefore, $q(1) = (-1)^n(1+x)^n$; $q(-1) = (-1)^n(x-1)^n$.

Hence $|SD(K_n{}^{\psi}) - xI| = \frac{(-1)^n(x-1)^n + (-1)^n(x+1)^n}{2} = (1-x)^n + (-1)^n(1+x)^n$.

The eigen values of $K_n{}^{\psi}$ are given by

$|SD(K_n{}^{\psi}) - xI| = 0$
$(-1)^n(x-1)^n + (-1)^n(x+1)^n = 0$
$(x-1)^n + (x+1)^n = 0.$

Therefore $\frac{(x-1)^n}{(x+1)^n} = -1 = cos\pi + isin\pi$

$$\left(\frac{x-1}{x+1}\right)^n = cos\pi + isin\pi$$

$= cos(2k+1)\pi + isin(2k+1)\pi;\ k = 0,1,2,\ldots,(n-1)$

$\frac{x-1}{x+1} = cos(2k+1)\frac{\pi}{n} + isin(2k+1)\frac{\pi}{n};$
$k = 0,1,2,\ldots,(n-1)$

$= e^{i(2k+1)\frac{\pi}{n}};\quad k = 0,1,2,\ldots,(n-1)$

$x = \frac{1+e^{i(2k+1)\frac{\pi}{n}}}{1-e^{i(2k+1)\frac{\pi}{n}}};\ k = 0,1,2,\ldots,(n-1)$

$= \frac{2isin(2k+1)\frac{\pi}{n}}{2-2cos(2k+1)\frac{\pi}{n}}$

$= \frac{2i.2sin(2k+1)\frac{\pi}{2n}cos(2k+1)\frac{\pi}{2n}}{2.2sin^2(2k+1)\frac{\pi}{2n}}$

$= icot(2k+1)\frac{\pi}{2n};\ k = 0,1,2,\ldots,(n-1)$

Hence the skew distance eigen values of $K_n{}^{\psi}$ are $icot\frac{\pi}{2n}, icot\frac{3\pi}{2n}, icot\frac{5\pi}{2n}, \ldots, icot\frac{(2n-1)\pi}{2n}$.

Therefore

$$\begin{aligned} E_{SD}(K_n{}^{\psi}) &= \left|icot\frac{\pi}{2n}\right| + \left|icot\frac{3\pi}{2n}\right| + \cdots \left|icot\frac{(2n-1)\pi}{2n}\right| \\ &= cot\frac{\pi}{2n} + cot\frac{3\pi}{2n} + \cdots + cot\frac{(2n-1)\pi}{2n} \\ &= \sum_{k=1}^{n} cot(2k+1)\frac{\pi}{2n}. \end{aligned}$$

Example 3.11 *Here is an infinite family with the property that deleting any edge will decrease the skew distance energy.*

By theorem 3.10 [5], $E_{SD}(ST_n^{\sigma}) = 2\sqrt{(n-1)}$, $(n \geq 4)$. Consider the oriented graph $ST_n^{\sigma}\backslash(1,n)$. As $|SD(ST_n^{\sigma}\backslash(1,n)) - xI| = x^n + (n-2)x^{n-2}$, the skew distance eigen values of $ST_n^{\sigma}\backslash(1,n)$ are $+\sqrt{(n-2)}i$, $-\sqrt{(n-2)}i$ and 0 occur in $(n-2)$ times.

Therefore $E_{SD}(ST_n^{\sigma}\backslash(1,2)) = 2\sqrt{(n-2)} < 2\sqrt{(n-1)} = E_{SD}(ST_n^{\sigma})$.

Let $K_n{}^{\psi}$ be the oriented complete graph on n vertices with the orientations of all the arcs go from low labels to high labels.

Consider the oriented graph $K_n{}^{\psi}\backslash\{(1,2)\}$. As

$$SD(K_n{}^{\psi}\backslash\{(1,2)\}) = \begin{bmatrix} 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ -1 & -1 & 0 & 1 & \cdots & 1 & 1 \\ -1 & -1 & -1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & 0 & 1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & 0 \end{bmatrix}$$

$|SD(K_n{}^{\psi}\backslash\{(1,2)\}) - xI|$

$$= \begin{vmatrix} -x & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -x & 1 & 1 & \cdots & 1 & 1 \\ -1 & -1 & -x & 1 & \cdots & 1 & 1 \\ -1 & -1 & -1 & -x & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & -x & 1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -x \end{vmatrix}$$

Using the elementary row operations,
$R_i \rightarrow R_i - R_{i-1}, \forall i = n, n-1, \cdots 3,2$, we get

$|SD(K_n{}^{\psi}\backslash\{(1,2)\}) - xI|$

$$= \begin{vmatrix} -x & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ x & -x & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & x-1 & -x-1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & x-1 & -x-1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -x-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & x-1 & -x-1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & x-1 & -x-1 \end{vmatrix}$$

Using the elementary column operations,
$C_j \rightarrow C_j - C_{j-1}, \forall j = n, n-1, \cdots 3,2$, we get

$|SD(K_n{}^{\psi}\backslash\{(1,2)\}) - xI|$

$$= \begin{vmatrix} -x & x & 1 & 0 & \cdots & 0 & 0 & 0 \\ x & -2x & x & 0 & \cdots & 0 & 0 & 0 \\ -1 & x & -2x & x+1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & x-1 & -2x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -2x & x+1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & x-1 & -2x & x+1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & x-1 & -2x \end{vmatrix}$$

Using Laplace's Expansion method,

$$|SD(K_n{}^{\psi}\{(1,2)\}) - xI| =$$

$$\begin{vmatrix} -x & x & 1 \\ x & -2x & x \\ -1 & x & -2x \end{vmatrix} \begin{vmatrix} -2x & x+1 & 0 & \cdots & 0 & 0 & 0 \\ x-1 & -2x & x+1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2x & x+1 & 0 \\ 0 & 0 & 0 & \cdots & x-1 & -2x & x+1 \\ 0 & 0 & 0 & \cdots & 0 & x-1 & -2x \end{vmatrix}$$

$$+ \begin{vmatrix} -x & x & 1 \\ x & -2x & x \\ 0 & 0 & x-1 \end{vmatrix} (-1)^{1+2+4+1+2+3}$$

$$\begin{vmatrix} x+1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ x-1 & -2x & x+1 & \cdots & 0 & 0 & 0 \\ 0 & x-1 & -2x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2x & x+1 & 0 \\ 0 & 0 & 0 & \cdots & x-1 & -2x & x+1 \\ 0 & 0 & 0 & \cdots & 0 & x-1 & -2x \end{vmatrix}$$

By 0.9.10[6] the determinant of the Tridiagonal matrices,

$$= \begin{cases} (-x^3 - 2x)[(-1)]^{n-3}[(n-2)c_1x^{n-3} \\ +(n-2)c_3x^{n-5} + \cdots + (n-2)c_{n-3}x] \\ +(-1)(x^3 - x^2)(x+1)(-1)^{n-4}[(n-3)c_1x^{n-4} \\ +(n-3)c_3x^{n-6} + \cdots + (n-3)c_{n-3}] \quad if n-3 \ is odd \\ \\ (-x^3 - 2x)[(-1)]^{n-3}[(n-2)c_1x^{n-3} \\ +(n-2)c_3x^{n-5} + \cdots + (n-2)c_{n-2}] \\ +(-1)(x^3 - x^2)(x+1)(-1)^{n-4}[(n-3)c_1x^{n-4} \\ +(n-3)c_3x^{n-6} + \cdots + (n-3)c_{n-4}x] \quad if n-3 \ is even \end{cases}$$

$$= \begin{cases} (x^3 + 2x)[(n-2)c_1x^{n-3} + (n-2)c_3x^{n-5} + \cdots \\ +(n-2)c_{n-5}x^3 + (n-2)c_{n-3}x] \\ -(x^4 - x^2)[(n-3)c_1x^{n-4} + (n-3)c_3x^{n-6} + \cdots \\ +(n-3)c_{n-5}x^2 + (n-3)c_{n-3}] if n-3 \ is odd \\ \\ -(x^3 + 2x)[(n-2)c_1x^{n-3} + (n-2)c_3x^{n-5} + \cdots \\ +(n-2)c_{n-4}x^2 + (n-2)c_{n-2}] + \\ (x^4 - x^2)[(n-3)c_1x^{n-4} + (n-3)c_3x^{n-6} + \cdots \\ +(n-3)c_{n-6}x^3 + (n-3)c_{n-4}x] if n-3 \ is even \end{cases}$$

$$= \begin{cases} [(n-2)c_1 - (n-3)c_1]x^n + [(n-2)c_3 + 2(n-2)c_1 \\ +(n-3)c_1 - (n-3)c_3]x^{n-2} + [(n-2)c_5 + 2(n-2)c_3 \\ +(n-3)c_3 - (n-3)c_5]x^{n-4} + \cdots \\ +[(n-2)c_{n-3} + 2(n-2)c_{n-5} + (n-3)c_{n-5} \\ -(n-3)c_{n-3}]x^4 + [2(n-2)c_{n-3} + \\ (n-3)c_{n-3}]x^2 if n is even \\ \\ x^n + [-(n-2)c_3 + (n-3)c_3 - 2(n-2)c_1 \\ -(n-3)c_1]x^{n-2} + [-(n-2)c_5 + (n-3)c_5 - 2(n-2)c_3 \\ -(n-3)c_3]x^{n-4} + \cdots + \\ [-(n-2)c_{n-2} - 2(n-2)c_{n-4} + (n-3)c_{n-4}]x^3 \\ -2(n-2)c_{n-2}x \quad if n is odd \end{cases}$$

$$= \begin{cases} x^n + \dfrac{(n-2)(n+1)}{2!}x^{n-2} \\ + \dfrac{(n-2)(n-3)(n-4)(n+3)}{4!}x^{n-4} + \cdots \\ + \dfrac{(n-2)(n-3)(2n-5)}{3!}x^4 + \dfrac{2n-3}{1!}x^2 \quad if \ n \ is even \\ \\ -x^n - \dfrac{(n-2)(n+1)}{2!}x^{n-2} \\ + \dfrac{(n-2)(n-3)(n-4)(n+3)}{4!}x^{n-4} + \cdots \\ +[-(n-2)(n-2)x^3] - 2x if n is odd \end{cases}$$

Theorem 3.12 *The skew distance spectrum of $K_{m,n}^{\sigma}$, σ - a canonicalorientation of $K_{m,n}$ is*

$$\begin{bmatrix} 1 & m+n-2 & 1 \\ -\sqrt{mni} & 0 & +\sqrt{mni} \end{bmatrix}$$

and hence $E_{SD}(K_{m,n}^{\sigma}) = 2\sqrt{mn}$

Proof. $K_{m,n}^{\sigma}$ is

$$|SD(K_{m,n}^{\sigma}) - xI|$$

$$= \begin{vmatrix} -x & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -x & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & -x & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & -x & 1 & 1 & \cdots & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -x & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & -x & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & -x & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 & -x \end{vmatrix}$$

Using the elementary row operations $R_{m+i} \to R_{m+i} - R_{m+1}, \forall i = 2,3,\cdots,n$, we get

$$\begin{vmatrix} -x & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -x & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & -x & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & -x & 1 & 1 & \cdots & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -x & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & x & -x & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x & 0 & \cdots & -x & 0 \\ 0 & 0 & 0 & \cdots & 0 & x & 0 & \cdots & 0 & -x \end{vmatrix}$$

Using the elementary column operations $C_{m+j} \to C_{m+j} - C_{m+1}$, we get

$$\begin{vmatrix} -x & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -x & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -x & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & -x & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & -x & x & \cdots & x & x \\ 0 & 0 & 0 & \cdots & 0 & x & -2x & \cdots & -x & -x \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x & -x & \cdots & -2x & -x \\ 0 & 0 & 0 & \cdots & 0 & x & -x & \cdots & -x & -2x \end{vmatrix}$$

Using Laplace's expansion method,

$|SD(K_{m,n}^{\sigma}) - xI|$

$$= (-1)^m x^m \begin{vmatrix} -x & x & \cdots & x & x \\ x & -2x & \cdots & -x & -x \\ \vdots & \vdots & & \vdots & \vdots \\ x & -x & \cdots & -2x & -x \\ x & -x & \cdots & -x & -2x \end{vmatrix}_{n\times n} +$$

$$m(-1) \begin{vmatrix} -x & 0 & \cdots & 0 & 0 \\ 0 & -x & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -x & 0 \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix}_{m\times m} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ x & -2x & -x & \cdots & -x & -x \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x & -x & -x & \cdots & -2x & x \\ x & -x & -x & \cdots & -x & -2x \end{vmatrix}$$

$$= (-1)^m x^m x^n \begin{vmatrix} -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -2 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & -1 & -1 & \cdots & -2 & 1 \\ 1 & -1 & -1 & \cdots & -1 & -2 \end{vmatrix} +$$

$$m(-1)(-x)^{m-1}(-1) \begin{vmatrix} -2x & -x & \cdots & -x & -x \\ -x & -2x & \cdots & -x & -x \\ \vdots & \vdots & & \vdots & \vdots \\ -x & -x & \cdots & -2x & -x \\ -x & -x & \cdots & -x & -2x \end{vmatrix}$$

$$= (-1)^m x^{m+n} (-1)^n +$$

$$m(-1)^{m-1}(-1)^2 x^{m-1}(-x)^{n-1} \begin{vmatrix} -2 & -1 & \cdots & -1 & -1 \\ -1 & -2 & \cdots & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & -1 & \cdots & -1 & -2 \end{vmatrix}$$

$$= (-1)^{m+n} x^{m+n} + m(-1)^{m+n} x^m x^n n$$
$$= (-1)^{m+n} x^{m+n} + (-1)^{m+n} mn x^{m+n-2}$$
$$= (-1)^{m+n} x^{m+n-2} [x^2 + mn].$$

The skew distance eigen values of $K_{m,n}^{\sigma}$ are given by

$$(-1)^{m+n} x^{m+n-2} [x^2 + mn] = 0$$

$x^2 + mn = 0$ or $x^{m+n-2} = 0$
$x = \pm\sqrt{mn}i$ or $x = 0$ $(m+n-2)$times.

Therefore, the skew distance spectrum of $K_{m,n}^{\sigma}$ is

$$\begin{bmatrix} 1 & m+n-2 & 1 \\ -\sqrt{mn}i & 0 & +\sqrt{mn}i \end{bmatrix}$$

$E_{SD}(K_{m,n}^{\sigma}) = \sqrt{mn} + \sqrt{mn} = 2\sqrt{mn}.$

Corollary 3.13 *As* $E_{SD}(K_{m,n}^{\sigma}) = 2\sqrt{mn}$, $E_{SD}(K_{n,n}^{\sigma}) = 2\sqrt{nn} = 2n$.

Example 3.14 *Here is an infinite family with the property that deleting any edge will increase the skew distance energy.*

Consider the oriented graph $K_{n,n}^{\sigma}$ with a canonical orientation σ. Delete an edge $(1,(n+1))$ from $K_{n,n}^{\sigma}$, the skew distance characteristic equation is

$$\left|SD\left(K_{n,n}^{\sigma}\backslash(1,n+1)\right) - xI\right|$$

$$= \begin{vmatrix} -x & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 \\ 0 & -x & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & -x & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & -x & 1 & 1 & \cdots & 1 & 1 \\ 0 & -1 & -1 & \cdots & -1 & -x & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & -x & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & -x & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 & -x \end{vmatrix} \forall, n \geq 2$$

$$= \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix},$$

where

$$A_{11} = \begin{bmatrix} -x & 0 & 0 & \cdots & 0 \\ 0 & -x & 0 & \cdots & 0 \\ 0 & 0 & -x & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & -x \end{bmatrix}_{n\times n}$$

$$A_{12} = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}_{n\times n}$$

$$A_{21} = \begin{bmatrix} 0 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \cdots & -1 & -1 \\ \vdots & & & & \vdots \\ -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \cdots & -1 & -1 \end{bmatrix}_{n\times n}$$

$$A_{22} = \begin{bmatrix} -x & 0 & \cdots & 0 & 0 \\ 0 & -x & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & -x & 0 \\ 0 & 0 & \cdots & 0 & -x \end{bmatrix}_{n\times n}$$

As A_{11} commutes with A_{12}, $|SD(K_{n,n}^{\sigma}\backslash(1,n+1)) - xI| = |A_{11}A_{22} - A_{21}A_{12}|$. Therefore,

$|SD(K_{n,n}^{\sigma}\backslash(1,n+1)) - xI|$

$$= \begin{vmatrix} x^2+(n-1) & (n-1) & (n-1) & \cdots & (n-1) & (n-1) \\ (n-1) & x^2+n & n & \cdots & n & n \\ (n-1) & n & x^2+n\cdots & n & n & \\ \vdots & & & & & \vdots \\ (n-1) & n & n & \cdots & n & x^2+n \end{vmatrix}$$

Using the elementary row operations,
$R_n \to R_n - R_{n-1}; R_{n-1} \to R_{n-1} - R_{n-2}, \cdots R_2 \to R_2 - R_1$
we get

$$\begin{vmatrix} x^2+(n-1) & (n-1) & (n-1) & \cdots & (n-1) & (n-1) \\ -x^2 & x^2+1 & 1 & \cdots & 1 & 1 \\ 0 & -x^2 & x^2 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & x^2 & 0 \\ 0 & 0 & 0 & \cdots & -x^2 & x^2 \end{vmatrix}$$

Using the elementary column operations,

$C_n \to C_n - C_{n-1}; C_{n-1} \to C_{n-1} - C_{n-2}, \cdots C_2 \to C_2 - C_1$ we get

$$\begin{vmatrix} x^2+(n-1) & -x^2 & 0 & \cdots & 0 & 0 & 0 \\ -x^2 & 2x^2+1 & -x^2 & \cdots & 0 & 0 & 0 \\ 0 & -x^2 & 2x^2 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 2x^2 & -x^2 & 0 \\ 0 & 0 & 0 & \cdots & -x^2 & 2x^2 & -x^2 \\ 0 & 0 & 0 & \cdots & 0 & -x^2 & 2x^2 \end{vmatrix}$$

Using Laplace's expansion method

$$|SD(K_{n,n}^{\sigma}\backslash(1,n+1)) - xI| = \begin{vmatrix} x^2+(n-1) & -x^2 \\ -x^2 & 2x^2+1 \end{vmatrix} \begin{vmatrix} 2x^2 & -x^2 & \cdots & 0 & 0 \\ -x^2 & 2x^2 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & -x^2 & 2x^2 \end{vmatrix}$$
$$+ \begin{vmatrix} x^2+(n-1) & -x^2 \\ 0 & -x^2 \end{vmatrix}$$
$$\times (-1)^{1+3+1+2} \begin{vmatrix} -x^2 & 0 & 0 & \cdots & 0 & 0 \\ -x^2 & 2x^2 & -x^2 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 2x^2 & -x^2 \\ 0 & 0 & 0 & \cdots & -x^2 & 2x^2 \end{vmatrix}$$
$$= [x^4 + (2n-1)x^2 + (n-1)](n-1)x^{2(n-2)} + (-x)^2(x^2+(n-1))(-1)^7(-x)^2$$
$$= (n-1)x^{2n-4} + (n-1)(2n-1)x^{2n-4}x^2 + (n-1)(n-1)x^{2n-4} - (x^6 + (n-1)x^4)(n-2)x^{2n-6}$$
$$= (n-1)x^{2n} + (n-1)(2n-1)x^{2n-2} + (n-1)(n-1)x^{2n-4} - (n-2)x^{2n} - (n-1)(n-2)x^{2n-2}$$
$$= [(n-1)x^{2n} - (n-2)] + [(n-1)(2n-1) - (n-1)(n-2)]x^{2n-2} + [(n-1)(n-1)]x^{2n-4}$$
$$= x^{2n} + (n^2-1)x^{2n-2} + (n-1)^2x^{2n-4}.$$

The skew distance eigen values of $K_{n,n}^{\sigma}\backslash(1,n+1)$ are given by

$$x^{2n} + (n^2-1)x^{2n-2} + (n-1)^2x^{2n-4} = 0$$
$$x^{2n-4}[x^4 + (n^2-1)x^2 + (n-1)^2] = 0$$
$$x = 0 \quad (2n-4) \quad times \& x^4 + (n^2-1)x^2 + (n-1)^2 = 0 \quad for n \geq 3$$
$$x = 0 \quad (2n-4) \quad times \& x^4 + (n+1)(n-1)x^2 + (n-1)(n-1) = 0 \quad for n \geq 3$$
$$and x^4 + (n+1)x^2 + (n-1) = 0 \; for n = 2$$

Put $t = x^2$ in $x^4 + (n+1)(n-1)x^2 + (n-1)^2 = 0$, we get
$t^2 + (n+1)(n-1)t + (n-1)^2 = 0$. Therefore

$$t = \frac{-(n^2-1) \pm \sqrt{(n^2-1) - 4(n-1)^2}}{2}$$
$$= \frac{-(n^2-1) \pm \sqrt{n^4+1-2n^2-4n^2-4+8n}}{2}$$
$$i.e x^2 = \frac{-(n^2-1) \pm \sqrt{n^4-6n^2-4+8n-3}}{2}$$
$$= \frac{-(n^2-1) \pm (n-1)\sqrt{n^2+2n-3}}{2}$$

As $(n^2-1) \geq (n-1)\sqrt{n^2+2n-3}$ and $-(n^2-1)$ is negative, $\frac{-(n^2-1) \pm (n-1)\sqrt{n^2+2n-3}}{2}$ is negative. Therefore

$$x = \pm\left[\frac{-(n^2-1) \pm (n-1)\sqrt{n^2+2n-3}}{2}\right]^{\frac{1}{2}}$$
$$= \pm\left[\frac{(n^2-1) \mp (n-1)\sqrt{n^2+2n-3}}{2}\right]^{\frac{1}{2}} i$$

Hence

$$E_{SD}[K_{n,n}^{\sigma}\backslash(1,n+1)] = 2(n^2-1) > 2n = E_{SD}(K_{n,n}^{\sigma})$$

Definition 3.15 *A regular graph on $2n$ vertices of the degree $2n-2$ is called a cocktail party graph and is denoted by $CP(n)$. $CP(n)^{\psi}$ is an oriented cocktail party graph on $2n$ vertices with the orientations of all the arcs go from low labels to high labels.*

Theorem 3.16 *Let K_n^{ψ} be the oriented complete graph on n vertices with the orientations of all the arcs go from low labels to high labels. Let $CP(n)^{\psi}$ be the oriented cocktail party graph on $2n$ vertices with the orientations of all the arcs go from low labels to high labels. Then $E_{SD}[CP(n)^{\psi}] = 2E_{SD}[K_n^{\psi}]$*

Proof.

$|SD[CP(n)^{\psi}] - xI|$

$$= \begin{bmatrix} -x & 1 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & 1 \\ -1 & -x & 1 & \cdots & 1 & -1 & 0 & 1 & \cdots & 1 \\ -1 & -1 & -x & \cdots & 1 & -1 & -1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & -x & -1 & -1 & -1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & -x & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 & -1 & -x & 1 & \cdots & 1 \\ -1 & -1 & 0 & \cdots & 1 & -1 & -1 & -x & \cdots & 1 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ -1 & -1 & -1 & \cdots & 0 & -1 & -1 & -1 & \cdots & -x \end{bmatrix}$$
$$= \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

$$where A = \begin{bmatrix} -x & 1 & 1 & \cdots & 1 \\ -1 & -x & 1 & \cdots & 1 \\ -1 & -1 & -x & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & -x \end{bmatrix} and$$

$$B = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 \\ -1 & -1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \cdots & 0 \end{bmatrix}.$$

As $(i,j)^{th}$ entry of AB = $(i,j)^{th}$ entry of $BA = -x + 2j - 2i - n$, $AB = BA$. Therefore

$$|SD[CP(n)^{\psi}] - xI| = \begin{bmatrix} A & B \\ B & A \end{bmatrix} = |A^2 - B^2|.$$

Now,

A^2

$$= \begin{bmatrix} x^2-(n-1) & -2x-(n-2) & \cdots & -2x+(n-4) & -2x+(n \\ 2x-(n-2) & x^2-(n-1) & \cdots & -2x+(n-6) & -2x+(n \\ 2x-(n-4) & 2x-(n-2) & \cdots & -2x+(n-8) & -2x+(n \\ \vdots & & & \vdots & \\ 2x+(n-2) & 2x+(n-4) & \cdots & 2x-(n-2) & x^2-(n- \end{bmatrix}$$

$$B^2 = \begin{bmatrix} -(n-1) & -(n-2) & -(n-4) & \cdots & (n-4) & (n-2) \\ -(n-2) & -(n-1) & -(n-2) & \cdots & (n-6) & (n-4) \\ -(n-4) & -(n-2) & -(n-1) & \cdots & (n-8) & (n-6) \\ \vdots & & & & & \vdots \\ (n-2) & (n-4) & (n-6) & \cdots & -(n-2) & -(n-1) \end{bmatrix}$$

Therefore

$|SD[CP(n)^{\psi}] - xI| = |A^2 - B^2|$

$$= \begin{vmatrix} x^2 & -2x & -2x & \cdots & -2x & -2x \\ 2x & x^2 & 2x & \cdots & -2x & -2x \\ 2x & 2x & x^2 & \cdots & -2x & -2x \\ \vdots & & & & & \vdots \\ 2x & 2x & 2x & \cdots & x^2 & -2x \\ 2x & 2x & 2x & \cdots & 2x & x^2 \end{vmatrix}$$

$$= x^n \begin{vmatrix} x & -2 & -2 & \cdots & -2 & -2 \\ 2 & x & 2 & \cdots & -2 & -2 \\ 2 & 2 & x & \cdots & -2 & -2 \\ \vdots & & & & & \vdots \\ 2 & 2 & 2 & \cdots & x & -2 \\ 2 & 2 & 2 & \cdots & 2 & x \end{vmatrix}$$

By 1.2 P_{16} in [6],

$$|SD[CP(n)^{\psi}] - xI| = x^n \left[\frac{(x-2)^n + (x+2)^n}{2}\right]$$

The skew distance eigen values of $CP(n)^{\psi}$ are given by

$$x^n \left[\frac{(x-2)^n + (x+2)^n}{2}\right] = 0$$

$\Rightarrow x = 0$ (n times) and $(x-2)^n + (x+2)^n = 0$. Thereforet the skew distance eigen values of $CP(n)^{\psi}$ are $0,0,\cdots,0$ (n times) $2icot\frac{\pi}{2n}, 2icot\frac{3\pi}{2n}, \cdots, 2icot\frac{2n-1}{2n}\pi$. Hence

$$E_{SD}[CP(n)^{\psi}] = 2\left[cot\frac{\pi}{2n} + cot\frac{3\pi}{2n} + \cdots + cot\frac{2n-1}{2n}\pi\right]$$
$$= 2E_{SD}(K_n^{\psi})$$
$$= E_{SD}(K_n^{\psi} \oplus K_n^{\psi}).$$

Example 3.17 *Here is an infinite family with the property that deleting all the edges in the cutset does not change the energy.*

Consider $CP(n^{\psi})$, the oriented cocktail party graph on $2n$ vertices with the orientations of all the arcs go from low labels to high labels. Then $E_{SD}[CP(n)^{\psi} \backslash E] = 2E_{SD}(K_n^{\psi}) = E_{SD}(K_n^{\psi} \oplus K_n^{\psi})$, where E is the set of all edges in the cutset. Let $E = \{(u_i, v_j)/i = 1,2,\cdots,n, j = 1,2,\cdots,n, i \neq j\}$. Edges in E are oriented with the orientations of all the arcs go from low labels to high labels.

As E is the cutset in $CP(n)^{\psi}$,

$$CP(n)^{\psi} \backslash E = K_n^{\psi} \oplus K_n^{\psi}$$
$$\therefore E_{SD}(CP(n)^{\psi} \backslash E) = E_{SD}(K_n^{\psi} \oplus K_n^{\psi})$$
$$= 2E_{SD}(K_n^{\psi})$$

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